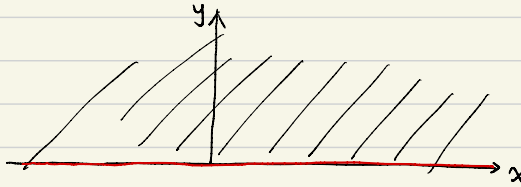


Fourier Analysis

April 19, 2022

Review.

Application 2: Steady state heat equation on the upper half plane.



$u = u(x, y)$ temperature distribution

$$\begin{cases} \Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, & x \in \mathbb{R}, y > 0, \\ u(x, 0) = f(x), & x \in \mathbb{R}. \end{cases}$$

Thm. Let $f \in \mathcal{S}(\mathbb{R})$. Let

$$P_y(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}, \quad y > 0.$$

$$u(x, y) = f * P_y(x), \quad x \in \mathbb{R}, y > 0.$$

Then u satisfies the following:

- ① $u \in C^2(\mathbb{R} \times \mathbb{R}_+)$ and $\Delta u = 0$
- ② $u(x, y) \Rightarrow f(x)$ as $y \rightarrow 0$
- ③ $\int_{-\infty}^{\infty} |u(x, y) - f(x)|^2 dx \rightarrow 0$ as $y \rightarrow 0$
- ④ $u(x, y) \rightarrow 0$ as $|x| + y \rightarrow \infty$.
("u vanishes at ∞ ")

Thm 1. (Uniqueness)

Let $u \in C^2(\mathbb{R} \times \mathbb{R}_+) \cap C(\overline{\mathbb{R} \times \mathbb{R}_+})$.

Suppose u satisfies that

$$\begin{cases} \Delta u = 0 & \text{on } \mathbb{R} \times \mathbb{R}_+ \\ u(x, 0) = 0 & \text{on } \mathbb{R} \end{cases}$$

Moreover suppose that $u(x, y) \rightarrow 0$ as $|x| + y \rightarrow \infty$.

Then $u(x, y) \equiv 0$ on $\mathbb{R} \times \mathbb{R}_+$.

Remark: The condition that u "vanishes" at infinity can not be dropped.

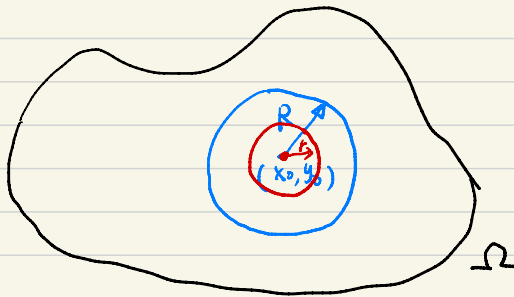
For instance, let $u(x, y) = y$. Then $\Delta u = 0$ and $u(x, 0) = 0$.

Lemma 2 (Mean Value property of harmonic function).

Let Ω be an open set of \mathbb{R}^2 . Let $u \in C^2(\Omega)$
and suppose $\Delta u = 0$ on Ω . Suppose

$B_R(x_0, y_0) \subset \Omega$ where

$$B_R(x_0, y_0) = \{ (x, y) \in \mathbb{R}^2 : (x-x_0)^2 + (y-y_0)^2 \leq R^2 \}$$



Then $\forall 0 < r < R$,

$$u(x_0, y_0) = \frac{1}{2\pi} \int_0^{2\pi} u(x_0 + r \cos \theta, y_0 + r \sin \theta) d\theta$$

Pf. Define

$$U(r, \theta) = u(x_0 + r \cos \theta, y_0 + r \sin \theta),$$

$$\text{for } 0 \leq r < R, \quad 0 \leq \theta < 2\pi.$$

Since $\Delta U = 0$, it follows that

$$\Delta U = 0 \quad \text{on } 0 \leq r < R, \quad 0 \leq \theta < 2\pi$$

Recall that

$$\Delta U = \frac{\partial^2 U}{\partial r^2} + \frac{1}{r} \frac{\partial U}{\partial r} + \frac{1}{r^2} \frac{\partial^2 U}{\partial \theta^2}$$

Hence

$$r^2 \frac{\partial^2 U}{\partial r^2} + r \frac{\partial U}{\partial r} + \frac{\partial^2 U}{\partial \theta^2} = 0$$

$$\text{so } r \frac{d}{dr} \left(r \frac{\partial U}{\partial r} \right) + \frac{\partial^2 U}{\partial \theta^2} = 0$$

$$\text{Define } f(r) = \frac{1}{2\pi} \int_0^{2\pi} U(r, \theta) d\theta$$

$$\begin{aligned} r \frac{d}{dr} \left(r \frac{df(r)}{dr} \right) &= \frac{1}{2\pi} \int_0^{2\pi} r \frac{\partial}{\partial r} \left(r \frac{\partial U}{\partial r} \right) d\theta \\ &= -\frac{1}{2\pi} \int_0^{2\pi} \frac{\partial^2 U}{\partial \theta^2} d\theta \end{aligned}$$

$$= -\frac{1}{2\pi} \left. \frac{\partial U}{\partial \theta} \right|_0^{2\pi}$$

$$= 0 \quad \left(\text{since } \frac{\partial U}{\partial \theta} \text{ is } 2\pi\text{-periodic in } \theta \right)$$

Hence

$$\frac{d}{dr} \left(r \frac{df}{dr} \right) \equiv 0 \quad \text{on } 0 < r < R$$

$$\text{So } r \frac{df}{dr} \equiv \text{Const} \quad \text{on } (0, R)$$

Notice that as $r \rightarrow 0$, $\frac{df}{dr}$ is bdd, hence

$$r \frac{df}{dr} = 0$$

It implies $f \equiv \text{const}$ on $0 < r < R$

By continuity $f(r) \equiv f(0)$.

That is

$$\frac{1}{2\pi} \int_0^{2\pi} \underbrace{U(r, \theta)}_{\substack{\downarrow \\ u(x_0 + r \cos \theta, y_0 + r \sin \theta)}} d\theta = \frac{1}{2\pi} \int_0^{2\pi} U(0, \theta) d\theta$$

$$= \frac{1}{2\pi} \int_0^{2\pi} u(x_0, y_0) d\theta$$

$$= u(x_0, y_0). \quad \square$$

Proof of Thm 1.

We use contradiction.

Suppose on the contrary that $U \neq 0$.

Then $\exists (x_0, y_0) \in \mathbb{R} \times \mathbb{R}_+$ such that

$$U(x_0, y_0) \neq 0.$$

WLOG, we assume that $U(x_0, y_0) > 0$

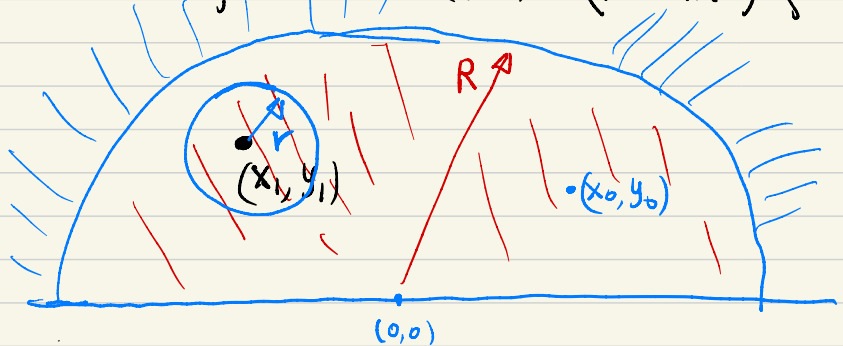
(otherwise, we may consider $-U$)

Since U vanishes at infinity, we can

find a large $R > 0$ such that

$$U(x, y) < \frac{1}{2} U(x_0, y_0)$$

for all $(x, y) \in (\mathbb{R} \times \mathbb{R}_+) \setminus B_R(0, 0)$



Notice that U is cts on

$$B_R(0,0) \cap \overline{\mathbb{R} \times \mathbb{R}_+} \quad (\text{closed, bdd})$$

So U attains its maximum

on $B_R(0,0) \cap \overline{\mathbb{R} \times \mathbb{R}_+}$ at some (x_1, y_1)

i.e.

$$U(x_1, y_1) = \sup_{\substack{(x,y) \\ \in B_R(0,0) \cap \overline{\mathbb{R} \times \mathbb{R}_+}}} U(x,y)$$

Since $U(x,y) < \frac{U(x_0, y_0)}{2} < U(x_1, y_1)$ for $(x,y) \in \overline{\mathbb{R} \times \mathbb{R}_+} \setminus B_R(0,0)$,

it follows that

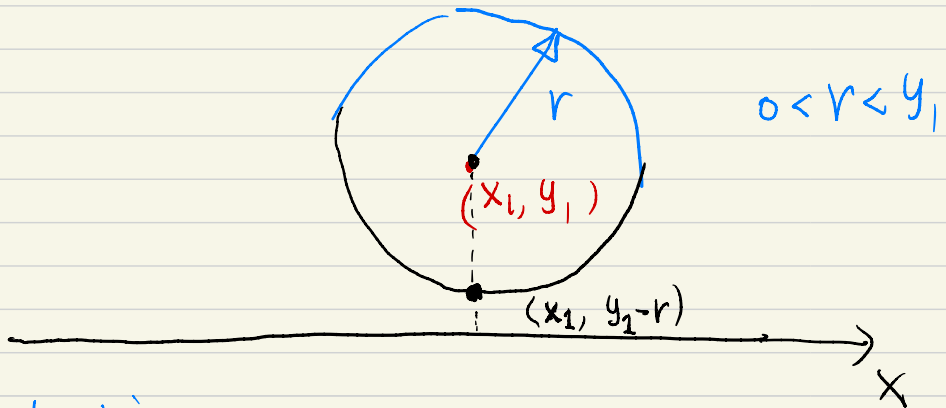
$$U(x_1, y_1) = \sup_{(x,y) \in \overline{\mathbb{R} \times \mathbb{R}_+}} U(x,y).$$

By the mean value property, for each

$$0 < r < y_1,$$

$$u(x_1, y_1) = \frac{1}{2\pi} \int_0^{2\pi} u(x_1 + r \cos \theta, y_1 + r \sin \theta) d\theta$$

$$\Rightarrow u(x_1 + r \cos \theta, y_1 + r \sin \theta) = u(x_1, y_1)$$



Letting $r \rightarrow y_1$,

$$u(x_1, 0) = \lim_{r \rightarrow y_1} u(x_1, y_1 - r)$$

$$= u(x_1, y_1) > 0.$$

Contradiction!



§ 5.3 Poisson summation formula.

Thm 3. (Poisson summation formula).

Let $f \in \mathcal{M}(\mathbb{R})$. Assume that $\hat{f} \in \mathcal{M}(\mathbb{R})$.

Let $F(x) = \sum_{n \in \mathbb{Z}} f(x+n)$.

Then

$$F(x) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}$$

In particular,

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

pf. Since $f \in M(\mathbb{R})$,

$$\sum_{n=-N}^N f(x+n) \Rightarrow F(x) \quad \text{on any compact set,} \\ \text{as } N \rightarrow \infty.$$

So F is cts on \mathbb{R} , and it is 1-periodic

Now let us calculate the Fourier coefficients of F .

$$\begin{aligned} \hat{F}(n) &= \int_0^1 F(x) e^{-2\pi i n x} dx \\ &= \int_0^1 \sum_{k=-\infty}^{\infty} f(x+k) \cdot e^{-2\pi i n x} dx \\ &= \sum_{k=-\infty}^{\infty} \int_0^1 f(x+k) e^{-2\pi i n x} dx \\ &= \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(y) e^{-2\pi i n (y-k)} dy \quad (\text{Letting } y = x+k) \end{aligned}$$

$$= \sum_{k=-\infty}^{\infty} \int_k^{k+1} f(y) \cdot e^{-2\pi i n y} dy$$

$$= \int_{-\infty}^{\infty} f(y) e^{-2\pi i n y} dy$$

$$= \hat{f}(n) \quad (\leftarrow \text{Fourier transform of } f)$$

That is, $\hat{F}(n) = \hat{f}(n)$, $n \in \mathbb{Z}$.

Since $\hat{f} \in M(\mathbb{R})$,

$$\sum_{n=-\infty}^{\infty} |\hat{f}(n)| < \infty$$

It follows that

$$\sum_{n=-\infty}^{\infty} |\hat{F}(n)| < \infty$$

Recall \hat{F} is the Fourier coefficient of F
on $[0, 1]$.

So

$$F(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x} \quad \text{on } [0, 1].$$

That is,

$$F(x) = \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x}.$$

□

Example 1. Let $f(x) = \frac{1}{\pi} \frac{y}{x^2 + y^2}$ on \mathbb{R} ,
where $y > 0$.

Recall that

$$\hat{f}\left(\frac{\xi}{y}\right) = e^{-2\pi|\xi|/y}.$$

Then $f, \hat{f} \in \mathcal{M}(\mathbb{R})$. By the Poisson Summation formula,

$$\begin{aligned} \sum_{n=-\infty}^{\infty} f(x+n) &= \sum_{n=-\infty}^{\infty} \hat{f}(n) e^{2\pi i n x} \\ &= \sum_{n \in \mathbb{Z}} e^{-2\pi|n|/y} e^{2\pi i n x} \end{aligned}$$

Letting $x=0$ gives

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{1}{\pi} \frac{y}{n^2+y^2} &= \sum_{n \in \mathbb{Z}} e^{-2\pi|n|y} \\ &= 1 + 2 \cdot \sum_{n=1}^{\infty} e^{-2\pi ny} \\ &= 1 + 2 \cdot \frac{e^{-2\pi y}}{1 - e^{-2\pi y}} \\ &= \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}}\end{aligned}$$

$$\Rightarrow \sum_{n=-\infty}^{\infty} \frac{1}{n^2+y^2} = \frac{\pi}{y} \cdot \frac{1 + e^{-2\pi y}}{1 - e^{-2\pi y}} \cdot$$

Example 2. Define the Theta function

$\Theta: \mathbb{R}_+ \rightarrow \mathbb{R}$ by

$$\Theta(s) = \sum_{n \in \mathbb{Z}} e^{-\pi n^2 s}.$$

Then Θ satisfies the following identity

$$\Theta(s) = \frac{1}{\sqrt{s}} \Theta\left(\frac{1}{s}\right).$$

Pf. Fix $s > 0$. Set $f(x) = e^{-\pi x^2 s}$.

Notice that

$$e^{-\pi x^2} \xrightarrow{\mathcal{F}} e^{-\pi \xi^2}.$$

So

$$\begin{aligned} f(x) &= e^{-\pi(\sqrt{s} \cdot x)^2} \xrightarrow{\sigma_{\mathcal{F}}} \frac{1}{\sqrt{s}} e^{-\pi \left(\frac{\xi}{\sqrt{s}}\right)^2} \\ &= \frac{1}{\sqrt{s}} e^{-\pi \xi^2 / s}. \end{aligned}$$

$$\text{That is, } \widehat{f}\left(\frac{\xi}{\sqrt{s}}\right) = \frac{1}{\sqrt{s}} \cdot e^{-\pi \xi^2 / s}.$$

Check: $f, \hat{f} \in \mathcal{M}(\mathbb{R})$.

By Poisson summation formula,

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n),$$

Hence

$$\sum_{n \in \mathbb{Z}} e^{-\pi n^2 s} = \sum_{n \in \mathbb{Z}} \frac{1}{\sqrt{s}} e^{-\pi \frac{n^2}{s}}$$

i.e.

$$\Theta(s) = \frac{1}{\sqrt{s}} \Theta\left(\frac{1}{s}\right). \quad \square$$